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THEORY OF RADIATION GENERATED BY CHARGED PARTICLES PASSING THROUGH CONTINIOUS PERIODIC MEDIUM

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16. Abstract

This article examines the problem of charge emission ina continuous periodic medium by means of a changed perturbation theory. General equations for an infinite periodic medium are given, as well as the solution of a homogeneous equation.

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THEORY OF RADIATION GENERATED BY CHARGED PARTICLES PASSING THROUGH CONTINIOUS PERIODIC MEDIUM

G.M. Garibian
and Yang Chi

I. Introduction

The dynamic phenomena arising during the diffraction of free x-rays in monocrystals (see, for example, [1] are well known. Similar phenomena may occur also for x-ray transition radiation, formed when ultrarelativistic charged particles pass through a crystal [2]. Physically, these phenomena are caused by the dynamic interaction of Bragg reflected and forward waves, and in principle they must occur each time that a wave is propagated in a sufficiently ideal periodic medium, with periods which are comparable with the wavelength. In particular, dynamic phenomena may occur in the long-wave regions of electromagnetic waves — for example, in the sub-millimeter or infrared region. It is characteristic that in each case the medium has an average dielectric constant which differs from unity, or is basically greater than unity. /3*

On the other hand, a study of a charged particle in a medium with continuous and periodically changing density was studied in [3-6]. In contrast to the similar problem in a system of plates which are regularly spaced with density which changes abruptly, when the solution of the problem may be obtained exactly (see, for example [7-9]), in the case of a continuous periodic medium a precise solution cannot be obtained. /4

*Numbers in the margin indicate pagination in the original foreign text.

This is due to the fact that in the latter case the problem is physically more general, and therefore the nonhomogeneous Hill equation arises of arbitrary form, whose explicit solution may be approximately given only when the medium density changes slightly. Thus, usually [3-6] the solution of the Hill equation is used as the zero approximation of the perturbation theory which corresponds to a forward wave in a medium, with allowance for the average value of the dielectric constant, whereas all the remaining waves are assumed to be small.

However, when the conditions of Bragg reflection are satisfied, a reflected wave will be of the same order, generally speaking, as a forward wave. Therefore, in this case the perturbation theory must be changed, so that both of these waves are considered in the zero approximation.

The present article examines the problem of charge emission in a continuous periodic medium by means of this changed perturbation theory. In the limiting cases, when emission frequency is far from a Bragg frequency, the formulas obtained change into the corresponding formulas of the regular perturbation theory. In addition, close to the Bragg frequencies the formulas obtained differ greatly from the well known formulas.

In particular, as a result of the dynamic interaction of a Bragg reflected and forward wave of coherently increased transitional emission formed by rigorously periodic inhomogeneities, close to the Bragg frequency very intense radiation arises. This radiation is almost monochromatic and is propagated both forward in the direction of the charge motion, and backwards at a very /5
small angle to the charge trajectory. The strength of this radiation at the spectral maxima not only greatly exceeds the strength of the transitional radiation in a periodic medium far from Bragg frequencies, which is produced in ordinary perturbation

theory [3-6], but also may be greater than the strength of Cherenkov radiation at the same frequencies (if the latter occurs).

In addition, this article also examines the case when a periodic medium has a finite length. Then, in addition to the radiation which is formed within the periodic medium, which emerges from it and is refracted, transitional radiation arises which is produced at the boundaries of the periodic medium and a vacuum.

The article [10] made the first experimental study of radiation of relativistic electron clusters in a waveguide with regularly spaced plates with a density which changes abruptly.

II. General equations for an infinite periodic medium

Let us assume that a rapid charged particle with a charge e moves uniformly along the z axis at the speed v in a medium whose dielectric constant is a periodic function of z with the period z_0:

$$\varepsilon = \varepsilon(z) = \varepsilon_0\left(1 + q\varphi(z)\right), \tag{1}$$

where $\varphi(z+z_0) = \varphi(z)$ and $\int_0^{z_0} \varphi(z)\,dz = 0$, i.e., ε_0 is the average dielectric constant of the medium.

If we represent the vectors of the electromagnetic fields in the form of Fourier integrals /6

$$\begin{aligned} \vec{D}(\vec{r},t) &= \int_{-\infty}^{\infty} \vec{D}(\vec{r},\omega)\exp(-i\omega t)\,d\omega \\ \vec{H}(\vec{r},t) &= \int_{-\infty}^{\infty} \vec{H}(\vec{r},\omega)\exp(-i\omega t)\,d\omega \end{aligned} \tag{2}$$

then we obtain the following from the Maxwell equation [5]

$$\begin{aligned} &\frac{d^2 Y(z)}{dz^2} + \left[\frac{\omega^2}{c^2}\varepsilon - \varkappa^2 + \frac{1}{2\varepsilon}\frac{d^2\varepsilon}{dz^2} - \frac{3}{4}\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\right)^2\right] Y(z) = \\ &= \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/2} \frac{e}{\pi v}\left(-\frac{i\omega v}{c^2} + \frac{i\omega}{v\varepsilon} - \frac{1}{\varepsilon^2}\frac{d\varepsilon}{dz}\right)\exp\left(i\frac{\omega}{v}z\right). \end{aligned} \tag{3}$$

The function Y(z) is connected with the Fourier components of the electromagnetic field vectors as follows

$$D_z(\vec{r},\omega)=\int_0^\infty(\varepsilon_0\varepsilon)^{1/2}Y(z)J_0(\varkappa\rho)\varkappa\,d\varkappa$$

$$D_\rho(\vec{r},\omega)=\int_0^\infty\left[\frac{e}{\pi v}\exp\left(i\frac{\omega}{v}z\right)-(\varepsilon_0\varepsilon)^{1/2}\left(\frac{Y(z)}{2\varepsilon}\frac{d\varepsilon}{dz}+\frac{dY(z)}{dz}\right)\right]J_1(\varkappa\rho)\,d\varkappa \tag{4}$$

$$H_\varphi(\vec{r},\omega)=\int_0^\infty\left[\frac{e}{\pi v}\exp\left(i\frac{\omega}{v}z\right)-i(\varepsilon_0\varepsilon)^{1/2}\frac{\omega}{c}Y(z)\right]J_1(\varkappa\rho)\,d\varkappa\ ,$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions of zero and first orders, χ, ρ are the transverse components of the wave vector $\vec{\chi}$ and the radius vector $\vec{r}$, respectively. We shall perform calculations for $\omega>0$, keeping the fact in mind that for $\omega<0$ the expressions for the field are obtained as complex conjugates.

Let us expand the function $\varphi(z)$ in Fourier series

$$\varphi(z)=\sum_{n=-\infty}^{\infty}a_{2n}\exp\left(2in\pi\frac{z}{z_0}\right)\ ,\quad a_0=0\ ,\ a_{-2n}=a^*_{2n} \tag{5}$$

We shall assume that the series (5) may be differentiated twice term by term. It is convenient to distinguish between two cases: (A) when the number of the Fourier amplitudes is infinite, they decrease monotonically with an increase in the number $|n|$ more slowly than n^{-3}; (B) when the number of the Fourier amplitudes a_{2n} is finite, and their value is arbitrary. In the case of a periodic medium consisting of plates, the corresponding Fourier series does not satisfy all conditions of case (A) namely, the Fourier amplitudes decrease as I/n. Nevertheless, as will be shown below, in case (A) the basic properties of the phenomenon occurring in a stack of plates [11] are qualitatively retained. /7

Substituting (1) and (5) in Equation (3), and introducing the dimensionless variable $\zeta=\pi z/z_0$, we obtain

$$\boxed{\frac{d^2Y}{d\zeta^2}+F(\zeta)Y=F_0(\zeta)}\ , \tag{6}$$

where $Y = Y(2\zeta/\pi)$ and

$$F(\zeta) = \sum_{n=-\infty}^{\infty} \theta_{2n} \exp(2in\zeta)$$

$$F_0(\zeta) = \frac{\omega^2 z_0^2}{v^2 q^2} \frac{ei}{q\omega z_0} \exp\left(i\frac{\omega z_0}{vq}\zeta\right) \sum_{n=-\infty}^{\infty} M_{2n} \exp(2in\zeta) \tag{7}$$

The Fourier coefficients θ_{2n} and M_{2n} are unequivocally determined when Equation (6) is derived. This equation is a non-homogeneous Hill equation. Following the general theory given in [12], we may formally write its solution. However, to obtain the explicit form of the solution, we shall assume [12] that the deviations of the medium from a homogeneous medium are small, i.e., $|q| \ll 1$. Then, within an accuracy of terms of a higher order of smallness, we have

$$\boxed{\begin{aligned} &\theta_0 = \left(\frac{\omega^2 \varepsilon_0}{c^2} - \varkappa^2\right)\frac{z_0^2}{\pi^2} \\ &\theta_{2n} = q\left(\frac{\omega^2 z_0^2 \varepsilon_0}{c^2 \pi^2} - 2n^2\right) a_{2n} \qquad (n \neq 0) \end{aligned}} \tag{8}$$

/8

$$M_0 = 1 - \beta^2 \varepsilon_0$$

$$M_{2n} = -q\left[\frac{1}{2}(1 + \beta^2 \varepsilon_0) + \frac{2\pi v}{\omega z_0} n\right] a_{2n} \quad (n \neq 0).$$

If the linearly independent solutions $y_1(\zeta)$ and $y_2(\zeta)$ of the homogeneous Hill equation (6) are known, then the solution of the non-homogeneous equation may be written in the form

$$Y = y_1(\zeta)[A_1 - V_1(\zeta)] + y_2(\zeta)[A_2 + V_2(\zeta)], \tag{9}$$

where

$$V_{1,2}(\zeta) = \frac{1}{W} \int_0^{\zeta} y_{2,1}(u) F_0(u)\, du,$$

$$W = y_1 y_2' - y_2 y_1'. \tag{10}$$

In order to determine the constants A_1, A_2 we shall use the condition that the solution of Equation (6) is periodic. (See [7]).

$$Y(z+z_0) = Y(z)\exp\left(i\frac{\omega z_0}{v}\right)$$

We thus obtain

$$A_{1,2} = \mp\frac{V_{1,2}(\pi)\exp\left[i\left(\mu_{1,2}\pi - \frac{\omega z_0}{v}\right)\right]}{1-\exp\left[i\left(\mu_{1,2}\pi - \frac{\omega z_0}{v}\right)\right]}, \tag{11}$$

where $\exp(i\mu_1\pi)$ and $\exp(i\mu_2\pi)$ are constants which, according to the Floquet theorem, occur before the solutions of the homogeneous Hill equation $y_1(\zeta)$ and $y_2(\zeta)$ when the arguments shift by the period π.

III. Solution of the homogeneous equation

According to the general solution of the homogeneous Hill equation, we assume /9

$$y(\zeta) = \exp(i\mu\zeta)\sum_{n=-\infty}^{\infty} C_{2n}\exp(2in\zeta) \tag{12}$$

Substituting (12) in the homogeneous equation which corresponds to Equation (6), we obtain the recurrence relationships

$$\left[\theta_0-(\mu+2n)^2\right]C_{2n} + {\sum_{m=-\infty}^{\infty}}' \theta_{2m}C_{2(n-m)} = 0, \tag{13}$$

where the prime over the sum sign means that a term with m = o must be omitted during the summation. These relationships give an infinite system of equations for determining the quantities C_{2n}, which are still unknown and which are Fourier coefficients of the solution (12) of the homogeneous Hill equation.

The coefficients $\theta_{2m}\ (m \neq 0)$ contain a small parameter q, and the series $\sum_m \theta_{2m}$ must be converging due to the assumption that the series (5) can be differentiated twice. It may be seen from Equation (13) that small coefficients θ_{2m} occur before all $C_{2(n-m)}$, whereas the coefficient $\theta_0-(\Gamma+2n)^2$ occurs before C_{2n}. If the latter quantity is not small for a single value of the whole number, n, then the system of Equation (13) has only a trivial solution. This system of equations may have a non-trivial solution only when $\theta_0-(\Gamma+2n)^2\approx 0$ for a certain whole number n. Since Γ is determined within an accuracy of an arbitrary odd number, without disturbing the generality, we may assume n = o in the last condition. Thus, the system of Equations (13) may have a non-trivial solution if $\Gamma\approx\pm\sqrt{\theta_0}$.

If the quantity $\sqrt{\theta_0}$ is not close to a whole number, then the quantities $\theta_0-(\Gamma+2n)^2$ cannot be small for an arbitrary whole number $n\neq 0$. Then it follows Equation (13) that in this case / 10
all the coefficients $C_{2n}\ (n\neq 0)$ are much less than C_0. Physically, this corresponds to the situation when the condition of Bragg reflection is not satisfied, and only one forward wave [6] is the main wave.

Let us now assume the quantity $\sqrt{\theta_0}$ is close to the whole number h >0. In the mathematical literature, this case has not been analyzed. (See [12]; the necessity of a special examination in this case was also indicated in [5]). Then among $C_{2n}\ (n\neq 0)$ one coefficient, namely C_{-2h}, will be of the same order as C_0, and the others remain small. Actually, relationship (13) assumes the following form in the case n = -h

$$[\theta_0-(\Gamma-2h)^2]C_{-2h}+\cdots+\theta_{-2h}C_0+\cdots=0$$

Since $\sqrt{\theta_0}\approx\Gamma\approx h$, then $\theta_0-(\Gamma-2h)^2$, and it thus follows that C_{-2h} and C_0 may be of the same order. Physically, this means that Bragg reflection occurs, and the reflected wave C_{-2h} plays an important role along with the forward wave C_0.

Keeping these statements in mind, we must retain the following two equations from System 13 as the zero approximation:

$$
\begin{aligned}
&(\theta_0-\Gamma^2)C_0+\theta_{2h}C_{-2h}=0\\
&\theta_{-2h}C_0+[\theta_0-(\Gamma-2h)^2]C_{-2h}=0 .
\end{aligned}
\tag{14}
$$

In order that Equation (14) have a non-trivial solution, it is necessary to require that the determinant of the system equal zero, i.e.,

$$
[\theta_0-\Gamma^2][\theta_0-(\Gamma-2h)^2]-\theta_{2h}\theta_{-2h}=0 \tag{15}
$$

The relationship obtained is an approximate characteristic equation for determining Γ, when the quantity $\sqrt{\theta_0}$ is close to the whole number h, which does not equal zero. /11

To solve Equation (15) we assume

$$
\theta_0=h^2+a \quad , \quad \Gamma=h+\delta , \tag{16}
$$

where $|a|\ll h^2$, $|\delta|\ll h$. Then for δ we obtained the two values

$$
\delta=\delta_{1,2}=\pm\frac{\sqrt{a^2-\theta_{2h}\theta_{-2h}}}{2h} \tag{17}
$$

We thus have

$$
C_{-2h}=\frac{2h\delta-a}{\theta_{2h}}C_0 \quad . \tag{18}
$$

Substituting $\Gamma_{1,2}=h+\delta_{1,2}$ in (12) instead of Γ, and taking into account (18), we obtain two linearly independent solutions of the zero approximation of the homogeneous Hill equation.

$$
\begin{aligned}
y^{(0)}_{1,2}(\zeta)=C_0\exp(i\delta_{1,2}\zeta)\Big[&\exp(ih\zeta)+\\
&+\frac{2h\delta_{1,2}-a}{\theta_{2h}}\exp(-ih\zeta)\Big]
\end{aligned}
\tag{19}
$$

Only two of the main terms C_0 and C_{-2h} were taken as the zero approximation from the entire sum (12). To obtain the subsequent approximation, it is necessary to take into account the contribution of the remaining terms C_{2n} to the solution. They may be determined by means of (13) in which it is sufficient to retain the two main components, m = n and n + h which correspond to C_0 and C_{-2h} in the summation over m. As a result, we obtain

$$\gamma_{12}^{(1)}(\zeta) = \gamma_{12}^{(0)}(\zeta) + C_0 \exp[i(h+\delta_{12})\zeta] \cdot$$

$$\cdot \sum_{n=-\infty}^{\infty}{}'' \frac{\theta_{2n}\theta_{2h} + (2h\delta_{12} - a)\theta_{2(n+h)}}{4\theta_{2h}\, n(h+n)} \exp(i2n\zeta) \tag{20}$$

where the double primes over the summation sign indicate /12
that we must omit terms with n = 0 and -h during the summation.

It may be readily seen that far from the Bragg frequencies when $|a| \gg |\theta_{2h}|$ and $|\theta_{-2h}|$, Formula 20 changes into the expression

$$\begin{aligned} y_1^{(1)}(\zeta) &\approx C_0 \exp(i\Gamma_1\zeta)\left\{1 + \sum_{n=-\infty}^{\infty}{}' \frac{\theta_{2n}}{4n(\Gamma_1+n)} \exp(2in\zeta)\right\} \\ y_2^{(1)}(\zeta) &\approx C_0' \exp(-i\Gamma_1\zeta)\left\{1 + \sum_{n=-\infty}^{\infty}{}' \frac{\theta_{-2n}}{4n(\Gamma_1+n)} \exp(-2in\zeta)\right\}. \end{aligned} \tag{21}$$

These formulas give the solution of the homogeneous equation for the problem being investigated, obtained as the first approximation of ordinary perturbation theory.

IV. Solution of the non-homogeneous equation

Having the solution of the homogeneous equation, using Formulas (9) - (11) we may write the solution of the non-homogeneous equation. Utilizing formula (20) after the appropriate computations we have

$$Y = Y^{(0)} + \Delta Y, \tag{22}$$

where $\Upsilon^{(0)}$ is obtained by means of the formula of the zero approximation (19) and

$$\Upsilon^{(0)}=\frac{i b_0^2 e}{4k^2\pi\omega\varepsilon_0}\sum_{n=-\infty}^{\infty}\left\{\frac{\theta_{2k}M_{2(n-k)}}{(b_n-k)^2-\delta_k^2}+\frac{4k^2(k^2-b_n^2-a)M_{2n}}{(b_n^2-k^2-\delta_k^2)^2-4k^2\delta_k^2}+\right.$$
$$\left.+\frac{\theta_{-2k}M_{2(n+k)}}{(b_n+k)^2-\delta_k^2}\right\}\exp(i b_n\xi). \tag{23}$$

Here $b_n=\frac{\omega z_0}{v\vartheta}+2n$. The quantity $\Delta\Upsilon$ occurs when taking into account corrections of the first approximation given in Formula 20, and has the form

$$\Delta\Upsilon=\frac{i b_0^2 e}{8k^2\pi\omega\varepsilon_0}\left\{\frac{a}{k^2}\sum_m\left[\frac{-\theta_{2k}M_{2(m-k)}+(2k(b_m-k)+a)M_{2m}}{(b_m-k)^2-\delta_k^2}+\right.\right.$$
$$\left.+\frac{-\theta_{-2k}M_{2(m+k)}+(2k(b_m+k)+a)M_{2m}}{(b_m+k)^2-\delta_k^2}\right]\exp(i b_m\xi)-$$
$$-\sum_m\sum_n{}^{'}\frac{1}{2n(k+n)}\left[\frac{-\theta_{2n}\theta_{2k}+(2k(b_{n+m}-k)+a)\theta_{2(n+k)}}{(b_{n+m}-k)^2-\delta_k^2}M_{2(m-k)}+\right. \tag{24}$$
$$+\frac{-\theta_{2(n+k)}\theta_{-2k}+(-2k(b_{n+m}+k)+a)\theta_{2n}}{(b_{n+m}+k)^2-\delta_k^2}M_{2m}+$$
$$+\frac{-\theta_{2n}\theta_{2k}+(2k(b_m-k)+a)\theta_{2(n+k)}}{(b_m-k)^2-\delta_k^2}M_{2(m-k)}+$$
$$\left.\left.+\frac{-\theta_{2(n+k)}\theta_{-2k}+(2k(b_m-k)+a)\theta_{2n}}{(b_m-k)^2-\delta_k^2}M_{2m}\right]\exp(i b_{n+m}\xi)\right\}.$$

Let us first examine the limiting case of a homogeneous medium when q = o. Then according to (8) the quantities θ_{2n} and M_{2n} equal zero in the case $n\neq 0$. As a result we obtain /13

$$\Upsilon=-\frac{i\omega e}{v^2\pi\varepsilon_0}\exp\left(i\frac{\omega}{v}z\right)\frac{1-\beta^2\varepsilon_0}{\omega^2/v^2-\omega^2\varepsilon_0/c^2+\varkappa^2} \tag{25}$$

Let us substitute this expression in (4) and let us integrate over $\varkappa$, using the following formulas (see, for example, [13]).

$$\int_0^\infty\frac{\varkappa J_0(\varkappa\rho)\,d\varkappa}{\varkappa^2-\eta^2}=K_0(-i\eta\rho)$$
$$\int_0^\infty\frac{J_1(\varkappa\rho)\,d\varkappa}{\varkappa^2-\eta^2}=-\frac{1}{\eta^2\rho}+\frac{K_1(-i\eta\rho)}{i\eta}, \tag{26}$$

where $K_{0,1}(x)$ — the modified Hankel function of zero and first orders, and $\mathrm{Im}\,\eta>0$. As a result of integration, we obtain, for example

$$D_z(\rho,\omega)=\frac{i\omega e}{v^2\pi}\exp\left(i\frac{\omega z}{v}\right)(\beta^2\varepsilon_0-1)K_0\left(-i\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon_0-1}\right). \tag{27}$$ /14

For a large absolute value of the argument, we have $K_{q_1}(x) \approx \sqrt{\pi/2x} \cdot \exp(-x)$. When $\beta^2 \varepsilon_0 > 1$, the field of the charge like the function ρ does not decrease exponentially, at large distances (in a transparent medium), which, as is known, corresponds to the formation of Cherenkov radiation.

When calculating integrals (4) using expressions (22) - (24), we must expand the latter into the simplest fractions of the type (25) with denominators containing $\varkappa^2$. The variable $\varkappa^2$ is only included by means of the quantities a and δ_1^2, in expressions (23) and (24). The quantity δ_1^2 in its turn is expressed by means of a according to formula (17), whereas from (8) and (16) we have

$$a = \frac{\omega^2 z_0^2 \varepsilon_0}{c^2 \gamma^2} - h^2 - \frac{\varkappa^2 z_0^2}{\gamma^2} \tag{28}$$

Taking these considerations into account, we may see that expressions (23) and (24) may be expanded into the simplest fractions with denominators of four types, $a \pm \alpha_+(n)$ and $a \pm \alpha_-(n)$, where

$$\alpha_\pm(n) = \sqrt{4h^2(b_n \pm h)^2 + \theta_{2k}\theta_{-2k}}\,, \tag{29}$$

where n is a corresponding whole number. For $\Upsilon^{(0)}$ we have

$$\begin{aligned} \Upsilon^{(0)} = \frac{i b_0^2 c}{2\pi\omega\varepsilon_0} \sum_{n=-\infty}^{\infty} \Bigg\{ & \frac{\theta_{2k} M_{2(n-k)}}{\alpha_-(n)} \left(\frac{1}{a+\alpha_-(n)} - \frac{1}{a-\alpha_-(n)} \right) + \\ & + \frac{h M_{2n}}{b_n} \Bigg[\frac{1}{\alpha_-(n)} \left(\frac{h^2 - b_n^2 + \alpha_-(n)}{a+\alpha_-(n)} - \frac{h^2 - b_n^2 - \alpha_-(n)}{a-\alpha_-(n)} \right) - \\ & - \frac{1}{\alpha_+(n)} \left(\frac{h^2 - b_n^2 + \alpha_+(n)}{a+\alpha_+(n)} - \frac{h^2 - b_n^2 - \alpha_+(n)}{a-\alpha_+(n)} \right) \Bigg] + \\ & + \frac{\theta_{-2k} M_{2(n+k)}}{\alpha_+(n)} \left(\frac{1}{a+\alpha_+(n)} - \frac{1}{a-\alpha_+(n)} \right) \Bigg\} \exp(i b_n \xi). \end{aligned} \tag{30}$$

Far from the Bragg frequencies, when $|a| \gg |\theta_{1l}|$ and $|\theta_{-1l}|$ /15
the solution Y derived above of the non-homogeneous equation changes, as it must, into the corresponding solution obtained by means of ordinary perturbation theory [3].

However, it must be emphasized that all of the formulas obtained are applicable under the conditions $|a| \ll k^2$ and $|\delta| \ll k$. If these conditions are not satisfied, instead of formulas (17) and (18), we shall have $\delta_{1l} = \pm\sqrt{2k^2 + a - \sqrt{4k^4 + 4k^2 a + \theta_{1l}\theta_{-1l}}}$ and $C_{-1l} = C_0(\delta^2 \pm 2k\delta - a)/\theta_{1l}$. The appropriate changes must be made in the subsequent formulas.

V. Radiation produced within an infinite continuous periodic medium.

Let us first establish the main contribution made to the quantity $D_\rho(\vec{r}, \omega)$. For this purpose, in formula (8), instead of Y, we must substitute expression (30) and disregard the term containing $d\varepsilon/dz$. We thus note that the expressions (23) and consequently (30) and (24) were obtained under the condition

$$\sqrt{\frac{\omega^2\varepsilon_0}{c^2} - \varkappa^2}\ \frac{z_0}{v} \approx k. \tag{31}$$

Condition (31) may be disturbed when integrating over $\varkappa$ in (4). In order to find under what conditions Equation (31) is not disturbed when integrating over $\varkappa$, we must find what values of $\varkappa$ make the main contribution to the integrals (4). Since the Bessel functions $J_{0,1}(\varkappa\rho)$ fluctuate and decrease for large values of the argument, the values of $\varkappa < \varkappa_0$ make the main contribution to the integrals (4) where $\varkappa_0 \sim \frac{1}{\rho}$. If it is required that $\varkappa_0 \ll \omega\sqrt{\varepsilon_0}/c$, we may then use formula (23) when calculating the integrals (4), if $\rho \gg c/\omega\sqrt{\varepsilon_0}$. It may be /16 readily seen that this condition may be readily satisfied. Then condition (31) may be written in form

$$\frac{\omega z_0}{c\pi}\sqrt{\varepsilon_0} \approx k. \tag{32}$$

which coincides with the Bragg condition in a medium with an average dielectric constant ε at an angle of incidence $\pi/2$. Replacing condition (31) by (32) means that we are confining ourselves to examining radiation emanating at small angles to be trajectory of the charge.

Keeping these statements in mind, after integration over $\varkappa'$ using formulas (26) we obtain the following expression

$$\begin{aligned} D_\rho^{(0)}(\vec{r},\omega) = {} & \frac{\exp(i\frac{3\pi}{4})}{2}\frac{\omega e}{v^2 z_0}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n=-\infty}^{\infty}\exp\left(i b_n \pi \frac{z}{z_0}\right) \cdot \\ & \cdot\left\{\frac{b_n\theta_{2k}M_{2(n-k)} + (k^2 - b_n^2 - \alpha_-(n))kM_{2n}}{\varkappa_{n1}^{3/2}\,\alpha_-(n)}\exp(i\varkappa_{n1}\rho) - \right. \\ & - \frac{b_n\theta_{2k}M_{2(n-k)} + (k^2 - b_n^2 + \alpha_-(n))kM_{2n}}{\varkappa_{n2}^{3/2}\,\alpha_-(n)}\exp(i\varkappa_{n2}\rho) + \\ & + \frac{b_n\theta_{-2k}M_{2(n+k)} - (k^2 - b_n^2 - \alpha_+(n))kM_{2n}}{\varkappa_{n3}^{3/2}\,\alpha_+(n)}\exp(i\varkappa_{n3}\rho) - \\ & \left. - \frac{b_n\theta_{-2k}M_{2(n+k)} - (k^2 - b_n^2 + \alpha_+(n))kM_{2n}}{\varkappa_{n4}^{3/2}\,\alpha_+(n)}\exp(i\varkappa_{n4}\rho)\right\}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \varkappa_{n1,2}^2 &= \frac{\omega^2}{c^2}\varepsilon_0 - \frac{k^2\pi^2}{z_0^2} \mp \frac{\pi^2}{z_0^2}\alpha_-(n) \\ \varkappa_{n3,4}^2 &= \frac{\omega^2}{c^2}\varepsilon_0 - \frac{k^2\pi^2}{z_0^2} \mp \frac{\pi^2}{z_0^2}\alpha_+(n). \end{aligned} \tag{34}$$

Making similar computations by means of formulas (4), (30) and (26) we obtain /17

$$\begin{aligned} D_z^{(0)}(\vec{r},\omega) = {} & \frac{\exp(i\frac{3\pi}{4})}{2}\frac{\omega e}{v^2 q}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{\exp\left(i b_n \pi \frac{z}{z_0}\right)}{b_n} \cdot \\ & \cdot\left\{\frac{b_n\theta_{2k}M_{2(n-k)} + (k^2 - b_n^2 - \alpha_-(n))kM_{2n}}{\varkappa_{n1}^{1/2}\,\alpha_-(n)}\exp(i\varkappa_{n1}\rho) - \right. \\ & - \frac{b_n\theta_{2k}M_{2(n-k)} + (k^2 - b_n^2 + \alpha_-(n))kM_{2n}}{\varkappa_{n2}^{1/2}\,\alpha_-(n)}\exp(i\varkappa_{n2}\rho) + \end{aligned} \tag{35}$$

$$+ \frac{b_n \theta_{-2k} M_{2(n+k)} - (k^2 - b_n^2 - \alpha_+(n)) k M_{2n}}{\varkappa_{n3}^{1/2}\, \alpha_+(n)} \exp(i\varkappa_{n3}\rho) - $$

$$- \frac{b_n \theta_{-2k} M_{2(n+k)} - (k^2 - b_n^2 + \alpha_+(n)) k M_{2n}}{\varkappa_{n4}^{1/2}\, \alpha_+(n)} \exp(i\varkappa_{n4}\rho) \Big\} . \qquad (35 \text{ Cont'd})$$

The quantity $H_\varphi(\vec{r},\omega)$ may be expressed by the formula obtained from (33) if we introduce the factor $\varepsilon_0\omega/\pi c b_n$ under the summation sign.

When Cherenkov radiation may occur in a medium, i.e. when $\beta^2\varepsilon_0 > 1$ and we are not close to the threshold, to the radiation described by formula (33), we must add the regular Cherenkov radiation in a medium with an average dielectric constant, which is determined by the formula (25) in the zero approximation. This is connected with the fact that the transverse component of the wave vector of the Cherenkov radiation $\omega\sqrt{\beta^2\varepsilon_0 - 1}/v$ far from the threshold is not small, and therefore ordinary Cherenkov radiation is not encompassed by formula (33) obtained when the condition (32) is satisfied. It may be seen from formula (33) that only those components for which the values of (34) are positive make a contribution to the field at large distances ρ. In addition, it may be seen from (33) that these components will be larger, the smaller are the values of (34). Since we are considering the case when the Bragg /18
condition (32) is satisfied, the values of (34) will be small only if $\alpha_\pm(n) \ll k^2$. It may be seen from the expressions for $\alpha_\pm(n)$ and b_n that this may be done if $|\omega\varepsilon_0/v\pi + 2n \pm k| \ll k$.

Keeping this in mind, we require that

$$\frac{\omega\varepsilon_0}{v\pi} = k' + d \quad . \qquad (36)$$

where $|d| \ll h'$, and h', is a positive whole number of the same parity as h. Substituting (36) in (29) and assuming that $n = n_0 + n_1$, when n_0 satisfies the conditions

$$2n_0 = \mp h - h' \tag{37}$$

for $\alpha_\pm(n)$ we obtain,

$$\alpha_\pm(n) = \sqrt{4h^2(d+2n_1)^2 + \theta_{\pm h}\theta_{-\pm h}}\,. \tag{38}$$

The order of magnitude of the quantity under the root sign in formula (38) is determined by the first term. Let us compare the quantities $\alpha_\pm(n_0)$, $\alpha_\pm(n_0 \pm 1)$, $\alpha_\pm(n_0 \pm 2)$. If h' is a small number, then the two quantities $\alpha_\pm(n_0)$ may always be less than the remaining quantities. If $h' \gg 1$, then these quantities are small and of the same order when $|n_1| \ll h'$.

It follows from conditions (36) and (32) — which lead to amplification of the radiation when they are satisfied — that the difference between the charge flight time and the average propagation time of the radiation during the nonhomogeneous period of the medium must equal the whole multiple of the radiation oscillation period.

We should note that if the quantity $|d| \gg \frac{\sqrt{\theta_{\pm h}\theta_{-\pm h}}}{h}$, in condition (36), it follows from formula (38) that $\alpha_\pm(n) \approx 2h|d+2n_1|$ and formula (33) changes into the corresponding formula obtained in ordinary pertubation theory.

Thus, if h' (and consequently h) is a small number, /19
i.e., if the wavelength of the radiation is comparable with the medium non-homogeneous period, the component with n = n_0 makes the basic contribution to formula (33). As a result, we obtain, for example

$$D_\rho^{(m)}(\vec{r},\omega) = -\frac{\exp\left[i\left(\frac{3}{4}+\frac{\ell d}{z_0}\right)\pi\right]}{2\alpha}\,\frac{\omega^2 e}{v^2 c}\left(\frac{\varepsilon_0}{2\rho\pi}\right)^{1/2}\cdot$$
$$\cdot\left\{\left[\frac{-(2hd+\alpha)M_{k-k'}+\theta_{2k}M_{-k-k'}}{\varkappa_1^{3/2}}\exp(i\varkappa_1\rho)+\right.\right.$$
$$\left.+\frac{(2hd-\alpha)M_{k-k'}-\theta_{2k}M_{-k-k'}}{\varkappa_2^{3/2}}\exp(i\varkappa_2\rho)\right]\exp\left(\frac{ih\pi z}{z_0}\right)-$$
$$-\left[\frac{\theta_{-2k}M_{k-k'}+(2hd-\alpha)M_{-k-k'}}{\varkappa_1^{3/2}}\exp(i\varkappa_1\rho)-\right. \qquad (39)$$
$$\left.\left.-\frac{\theta_{-2k}M_{k-k'}+(2hd+\alpha)M_{-k-k'}}{\varkappa_2^{3/2}}\exp(i\varkappa_2\rho)\right]\exp\left(-\frac{ih\pi z}{z_0}\right)\right\},$$

where

$$\alpha = \sqrt{4h^2d^2+\theta_{2k}\theta_{-2k}}$$
$$\varkappa_{1,2}^2 = \frac{\omega^2}{c^2}\varepsilon_0 - \frac{h^2\pi^2}{z_0^2} \mp \frac{\pi^2}{z_0^2}\alpha. \qquad (40)$$

We should note that in the particular case of ultra-relativistic particles, i.e. $1-\beta^2 \ll 1$, and in the region of frequencies where $\varepsilon_0 - 1 = \xi_0 < 0$ и $|\xi_0| \ll 1$, the expression obtained (39) must coincide with the expression $E_{\rho\omega}(\vec{r},\omega)$ for the case of Bragg refraction which is almost precisely forward; this expression was obtained in [2]. Actually, if we integrate over the wave vector $\vec{k}$ using formulas (29) from [2], we obtain an expression which coincides with (39), given in this particular case.

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Let us now find the strength of radiation arising per unit length of the trajectory of motion of a charged particle. For this purpose, let us calculate the flow of the Poynting vector passing through a circular region $\rho_1 \le \rho \le \rho_2$ in a plane perpendicular to the z axis. (see Figure 1).

$$dW = \frac{c}{4\pi}\int_{-\infty}^{\infty} dt \int_{\rho_1}^{\rho_2} 2\pi\rho\, d\rho\, E_\rho(\vec{r},t)\, H_\varphi(\vec{r},t). \qquad (41)$$

Thus $\rho_1 = z\,tg\,\vartheta$, $\rho_2 = (z - dz)\,tg\,\vartheta$, where $\vartheta = arctg(c\,Re\,\varkappa_1/\omega\sqrt{\varepsilon_0})$ is the angle of radiation. The quantities $E_\rho(\vec{r},t)$ and $H_\varphi(\vec{r},t)$ may be readily obtained from (30) and (2). When calculating the integral (41), interference terms occur which contain factors of the type $\exp i(\varkappa_1 - \varkappa_2)\rho$ or $\exp 2ih\pi z/z_0$, which must be omitted.

As a result, we obtained the following for radiation emanating in the forward direction

$$\frac{dW}{dz} = \frac{e^2}{4v^4}\int\frac{\omega^3}{|\alpha|^2\varepsilon_0}\left\{\frac{|-(2hd+\alpha)M_{h-h'} + \theta_{2h}M_{-h-h'}|^2}{|\varkappa_1|^3}Re\,\varkappa_1 + \frac{|(2hd-\alpha)M_{h-h'} - \theta_{2h}M_{-h-h'}|^2}{|\varkappa_2|^3}Re\,\varkappa_2\right\}d\omega, \tag{42}$$

and the following for radiation emanating backwards

$$\frac{dW}{dz} = \frac{e^2}{4v^4}\int\frac{\omega^3}{|\alpha|^2\varepsilon_0}\left\{\frac{|\theta_{-2h}M_{h-h'} + (2hd-\alpha)M_{-h-h'}|^2}{|\varkappa_1|^3}Re\,\varkappa_1 + \frac{|\theta_{-2h}M_{h-h'} + (2hd+\alpha)M_{-h-h'}|^2}{|\varkappa_2|^3}Re\,\varkappa_2\right\}d\omega. \tag{43}$$

It may be seen from formulas (42) and (43) that the radiation strength is determined by the Fourier amplitudes a_{2h}, $a_{2(h-h')}$ and $a_{-2(h+h')}$, independently of the presence or absence of other amplitudes.

Since we are considering the small numbers h and h', cases A and B, which were given in Section 2, do not differ in essence from each other. We must emphasize that the formulas are only valid in the vicinity of Bragg frequencies $\omega_B = h\pi c/z_0\sqrt{\varepsilon_0}$ and when $\omega > 0$. /21

Let us now analyze the expressions obtained for an arbitrary value of ε_0 separately for the cases of when $\beta^2\varepsilon_0$ differs significantly from unity (far from the Cherenkov radiation threshold) and when $\beta^2\varepsilon_0 \approx 1$ (close to the threshold). This division is due to the fact that in the first case, as was noted above, Cherenkov radiation is not encompassed by formula (39), whereas in the second case the Cherenkov radiation which is produced is propagated at a small angle and it automatically is taken into account by formula (39).

Let us introduce a small deviation of the frequency $\nu = (\omega - \omega_s)/\omega_s$. Then $d = k'(d_0 + \nu)$, where

$$d_0 = \frac{k}{k'\beta\sqrt{\varepsilon_0}} - 1 \tag{44}$$

and in addition we have the following from (40)

$$\varkappa_{1,2}^2 = \frac{2\pi^2 k^2}{\lambda_0^2}\left\{\nu \mp \frac{1}{k}\sqrt{k'^2(d_0+\nu)^2 + \frac{\theta_{2k}\theta_{-2k}}{4k^2}}\right\}. \tag{45}$$

Let us first examine the first case when $\beta^2\varepsilon_0 \not\approx 1$. It may be seen from (32) and (36) that in this case $k \neq k'$, and $k > k'$ corresponds to $\beta^2\varepsilon_0 > 1$, and $k < k'$ corresponds to $\beta^2\varepsilon_0 < 1$. With allowance for the absorbing capacity of the medium, i.e. if we assume that $\varepsilon_0 = \varepsilon_0' + i\varepsilon_0''$ $(|\varepsilon_0''| \ll |\varepsilon_0'|)$, from Equation (40) for $\varkappa_i^2$, we find that

$$\begin{aligned} Re\,\varkappa_i &= \frac{\pi}{\lambda_0}\sqrt{\frac{2k^2\nu \mp \alpha + \sqrt{(2k^2\nu \mp \alpha)^2 + k^4(\varepsilon_0''/\varepsilon_0')^2}}{2}} \\ Im\,\varkappa_i &= \frac{\pi^2 k^2 \varepsilon_0''}{2 Re\,\varkappa_i \lambda_0^2 \varepsilon_0'} \end{aligned} \tag{46}$$

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The denominators $|\varkappa_i|^3$ in Formulas (42) and (43), have the form

$$\pi^3\left[\left(2k^2\nu \mp \sqrt{4k^2 d^2 + \theta_{2k}\theta_{-2k}}\right)^2 + k^4(\varepsilon_0''/\varepsilon_0')^2\right]^{3/4}/\lambda_0^3.$$

It may thus be seen that the radiation strength (42) and (43) reaches a maximum when

$$2k^2\nu \mp \sqrt{4k^2 d^2 + \theta_{2k}\theta_{-2k}} \approx 0.$$

Expressing d by ϑ and d_0, we find that the maximum occurs when

$$\vartheta \approx \vartheta_0 = \frac{d_0 k' \pm \sqrt{k^2 k'^2 d_0^2 + (k^2 - k'^2)\theta_{2k}\theta_{-2k}/4k^2}}{k^2 - k'^2}$$

At the maximum of the radiation strength (42) and (43) are inversely proportional to $k^2 \varepsilon_0'' / \varepsilon_0'$.

Let us determine the ratio of the strength of radiation emanating forward at the maximum to the strength of Cherenkov radiation at the same frequency (if the latter occurs). This ratio has the order of magnitude

$$\left| q^2 a^2_{2(k-k')} [(\sqrt{\varepsilon_0'}+1)^2 - 2]^2 \omega_b^2 \varepsilon_0' \varepsilon_0^2 / 16\sqrt{2}\, \varepsilon_0''(\varepsilon_0'^2 - 1) c^2 v^2 \right.$$

(We assume that $|a_{-2(k+k')}| \ll |a_{2(k-k')}|$). For a sufficiently small absorption, i.e. if the quantity ε_0'' is sufficiently small, this ratio may be on the order of unity and greater.

We may establish the spectral width of the maximum as that deviation $\Delta\vartheta = \vartheta - \vartheta_0$, at which $\mathrm{Re}\, \varkappa_i \sim \mathrm{Im}\, \varkappa_i$. It may readily be seen that $\Delta\vartheta \sim k^2 \varepsilon_0'' / (k^2 - k'^2)\varepsilon_0'$. The radiation angle has the order $k\sqrt{|\varepsilon_0''/\varepsilon_0'|}$, i.e. it is very small for a weakly absorbing media.

For sufficient large values of $|\vartheta| \gg d_0$ and $\sqrt{\theta_{2k}\theta_{-2k}}/k^2$, the quantity $|\varkappa_i|^2$ will increase in proportion to $|\vartheta|$ and therefore the strength of the radiation described by formulas (42) and (43) will be small.

A similar situation occurs close to the Cherenkov threshold when $\beta^2 \varepsilon_0 \approx 1$, i.e. h = h'. The difference lies in the fact that in this case, close to each Bragg frequency ω_b, there is in all one maximum when /23

$$\vartheta_0 = -\frac{d_0}{2} - \frac{\theta_{2k}\,\theta_{-2k}}{8 d_0 k^4}.$$

The width of the maximum has the order $\varepsilon''_0/\varepsilon'_0$. Far from the Bragg frequency, the radiation emitted in a forward direction, just as in the case $k \neq k'$, rapidly decreases. With respect to the radiation which is emitted in the backward direction, above the threshold it changes into ordinary Cherenkov radiation, and below the threshold it becomes weak.

Thus, analyzing formulas (42) and (43), we find that when a charged particle passes through a periodic, slightly absorbing medium, the most interesting occurrence is that close to the Bragg frequency, very intense and almost monochromatic radiation arises, which is propagated both forward and backward with respect to the direction of the charge motion. Apart from this radiation, there is also Cherenkov radiation (if $\beta^2\varepsilon_0 > 1$) and the regular weak transitional radiation far from the Bragg frequencies, which is produced by the inhomogeneities of the medium and is described by formulas obtained in the regular pertubation theory [3-6], which may also be obtained from formula (33) in this study. The nature of the intense and almost monochromatic radiation close to the Bragg frequencies is a result of dynamic interaction between Bragg reflected and forward waves, of coherently intensified transitional radiation produced by the rigorously periodic inhomogeneities of the medium. Since the inhomogeneities of the medium are continuously distributed over the entire trajectory of motion of the charge, the strength of the transitional radiation is proportional to the length of the trajectory, both /24
close to and far from the Bragg frequencies.

By way of illustration, Figures 2 and 3 show curves of the spectral dependence of the number of quanta $dN/d\nu$, emitted from a trajectory one radiation wavelength long. Thus, Figure 2 pertains to radiation emitted in the forward direction and Figure 3 pertains to radiation emitted in the backward direction.

These curves were calculated using formulas (42) and (43), in which it is assumed that only the Fourier amplitudes $a_{1(k-1)}$ and a_{1k} differ from zero when $k = 5$ and $k' = 3$. In addition, $\varepsilon_0' = 2.778$, $\varepsilon_0''/\varepsilon_0' = 3\cdot 10^{-4}$, $(1-\beta^2)/2 = 1\cdot 10^{-4}$, $q = 0.15$. The numbers 1, 2 and 3 mean that A_4 and A_{10} respectively equal 0.25 and 0.25; 0.45 and 0.05; 0.4995 and 0.0005. The dashed horizontal line in Figure 2 pertains to Cherenkov radiation.

Let us now assume that h and h' are much greater than unity, i.e., the radiation wavelength is much less than the medium inhomogeneity period.

Let us first consider case A which was pointed out in section 2. In addition to the terms (39) in formula (33), a large contribution was also made by terms with $n = n_0 + n_1$, where $n_1 = \pm 1, \pm 2, \ldots$, and $|n_1| \ll k'$. Let us write the explicit form of these terms for the particular case of ultrarelativistic particles $(1-\beta^2 \ll 1)$ and in that frequency region where $|\varepsilon_0 - 1| = |g_0| \ll 1$. In this case, as may be seen from conditions (32) and (36) $k = k'$. Due to the fact that $k \gg 1$, in formula (38), we may disregard the term $\theta_{1l}\theta_{-1k}$, as compared with the term $4k^2(d+2n_1)^2$, excluding certain values of ν, for which $d + 2n_1 = 0$. Since $d = k(1-\beta+\nu-g_0/2)$, we have

$$\varkappa_{ni}^2 \approx \frac{2k^2\pi^2}{a^2}\left\{\nu \mp \left|1-\beta-\frac{g_0}{2}+\nu+\frac{2n_1}{k}\right|\right\} \tag{47}$$

where $n = n_0 + n_1$, and the signs $\pm$ correspond to $i = 1.3$ and $i = 2.4$.

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We readily see from the expression for b_n, that $b_n = k + d + 2n_1$ for the first two terms of formula (33) and $b_n = -k + d + 2n_1$ for the last two terms. This means that the first two terms correspond to waves which are propagated forward in the direction of motion of the charge, and the last two terms — in the opposite direction.

Keeping the fact in mind that when $h \gg n_1$ the quantity M_{-2h+2n_1}, is much less than M_{2n_1}, we find from formula (33) that part of the wave $D_\rho(\vec{r},\omega)$, which is propagated in the forward direction, has the form

$$\frac{\exp(i\frac{3\pi}{4})}{2}\,\frac{\omega e}{v^2 z_0}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n_1}\exp\left[(h+d+2n_1)\frac{\pi z}{z_0}\right]\cdot \frac{2h\,M_{2n_1}}{\varkappa_{n_j}^{1/2}}\exp(i\varkappa_{n_j}\rho), \tag{48}$$

where j = 1 or 2 depending on whether the quantity $d+2n_1$. is positive or negative, and summation is performed over the whole numbers n_1, such that $\varkappa_{n_j}^2 > 0$. It may be readily seen from formula (47) that independently of the sign of d + $2n_1$ we have

$$\varkappa_{n_j}^2 = -\frac{h^2\pi^2}{z_0^2}\left(1-\beta^2-g_0-\frac{4n_1}{h}\right). \tag{49}$$

If we introduce the angle of radiation $\vartheta = Re\,\varkappa_{n_j} c/\omega$, from formula (49) we obtain

$$\vartheta^2 = -\frac{4\pi n_1 c}{\omega z_0} - (1-\beta^2) + g_0. \tag{50}$$

The radiation described by formula (48) is regular transitional radiation formed by periodic inhomogeneities of the medium, when the wavelength is much less than the inhomogeneity period. Since in this case $|\alpha| \gg |\theta_{2l}|$, according to the statements made at the end of Section 3,this radiation may be calculated in the "single wave" perturbation theory [3-6]. In particular, the radiation angles (50) coincides with the angles obtained in [4] (see also [14]). /26

With respect to a wave which is propagated forward, as may be seen from (33), due to the smallness of θ_{-2l}, and M_{-2h+2n_1}, it is very weak in this case.

In Case B, since the Fourier amplitudes A_{2n} are arbitrary, we must calculate the radiation by the method given above of the "two-wave" theory, if the quantity $\theta_{2\ell}\theta_{-2\ell}$ is sufficiently large. [See Formulas (33) - (35)].

V. Radiation of a finite periodic medium.

Let us assume a periodic medium is finite and is located, for example, between the two planes z = 0 and $z = \ell = Nz_0$, where N is a whole number, and there is a vacuum outside of it. The radiation outside of the periodic medium may be obtained by using the condition that the fields are continuous at the boundaries of the medium and the vacuum, and by using the results given in the preceding section.

For this purpose, we should note that the transverse component of the field $E_\rho(\vec{r},\omega)$ outside of the periodic medium may be written in the form

$$\begin{aligned} E_\rho(\vec{r},\omega) &= \int_0^\infty \left\{ \frac{e}{\pi v} \frac{\varkappa^2 \exp\left(i\frac{\omega z}{v}\right)}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + G_1 \exp(-i\lambda_0 z_0) \right\} J_1(\varkappa\rho)\, d\varkappa \\ E_\rho(\vec{r},\omega) &= \int_0^\infty \left\{ \frac{e}{\pi v} \frac{\varkappa^2 \exp\left(i\frac{\omega z}{v}\right)}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + G_2 \exp(i\lambda_0 z_0) \right\} J_1(\varkappa\rho)\, d\varkappa \end{aligned} \tag{51}$$

for the regions $z<0$ and $z>\ell$, respectively. Thus G_1 and G_2 are arbitrary constants, and $\lambda_0^2 = \omega^2/c^2 - \varkappa^2$. Within the periodic medium, the waves obtained in the preceding section must be 27
supplemented by the free fields, which arise due to the presence of the boundaries, and which are caused by the solution of the homogeneous Hill equation. Taking equations (16) - (18) into account, these fields may be represented in the form

$$\begin{aligned} E_\rho^{cl}(\vec{r},\omega) = -\int_0^\infty \Big\{ & G_3 \left\{ \exp\left[i(k+\delta_1)\frac{\pi z}{z_0}\right] - \frac{2k\delta_1 - a}{\theta_{2\ell}} \exp\left[i(-k+\delta_1)\frac{\pi z}{z_0}\right] \right\} + \\ & + G_4 \left\{ \exp\left[i(k-\delta_1)\frac{\pi z}{z_0}\right] + \frac{2k\delta_1 + a}{\theta_{2\ell}} \exp\left[-i(k+\delta_1)\frac{\pi z}{z_0}\right] \right\} \Big\} J_1(\varkappa\rho)\, d\varkappa . \end{aligned} \tag{52}$$

The arbitrary constants $G_1, \dots, G_4$ are determined from the matching of the corresponding fields at the boundaries z = 0 and z = 1:

$$
\begin{aligned}
&G_1 + p_1 G_3 + p_2 G_4 = q_1 \\
&\frac{k\pi}{\lambda z_0} G_1 - \varepsilon_0 p_3 G_3 - \varepsilon_0 p_4 G_4 = q_2 \\
&\exp\left[i\left(\lambda_0 - \frac{k\pi}{z_0}\right)l\right] G_2 + p_1 \exp\left(i\frac{\pi l \delta_1}{z_0}\right) G_3 + p_2 \exp\left(-i\frac{\pi l \delta_1}{z_0}\right) G_4 = q_1 \exp\left[i\left(\frac{\omega}{v} - \frac{k\pi}{z_0}\right)l\right] \\
&-\frac{k\pi}{\omega\lambda_0} \exp\left[i\left(\lambda_0 - \frac{k\pi}{z_0}\right)l\right] G_2 - \varepsilon_0 p_3 \exp\left(i\frac{\pi l \delta_1}{z_0}\right) G_3 - \varepsilon_0 p_4 \exp\left(-i\frac{\pi l \delta_1}{z_0}\right) G_4 = q_2 \exp\left[i\left(\frac{\omega}{v} - \frac{k\pi}{z_0}\right)l\right]
\end{aligned}
\tag{53}
$$

Here we have introduced the notation

$$
\begin{aligned}
p_1 &= \frac{\theta_{2k} - 2k\delta_1 + a}{\theta_{2k}}, \quad p_2 = \frac{\theta_{2k} + 2k\delta_1 + a}{\theta_{2k}} \\
p_3 &= \frac{\theta_{2k} + 2k\delta_1 - a}{\theta_{2k}}, \quad p_4 = \frac{\theta_{2k} - 2k\delta_1 - a}{\theta_{2k}}
\end{aligned}
\tag{54}
$$

With respect to the quantities q_1 and q_2, in the right side of the equations (53), we may obtain their explicit form only for the most interesting case when the radiation wavelength is on the order of the medium inhomogeneity period: /28

$$
\begin{aligned}
q_1 &= -\frac{e}{v\tau}\left[\frac{\varkappa^2}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} - \frac{4(d^2-\delta_1^2) + M_{k-\varepsilon}(-2kd - a - \theta_{-2k}) + M_{-k-\varepsilon}(-2kd + a + \theta_{2k})}{4\varepsilon_0(d^2-\delta_1^2)}\right] \\
q_2 &= -\frac{ek}{\omega\varepsilon_0}\left[\frac{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + \frac{M_{k-\varepsilon}(-2kd - a + \theta_{-2k}) + M_{-k-\varepsilon}(2kd - a + \theta_{2k})}{4(d^2-\delta_1^2)}\right].
\end{aligned}
\tag{55}
$$

Solving the system of equations (53) we may obtain the explicit form of the constants

$$
\begin{aligned}
G_1 = \Big\{&\left(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 z_0}{k\pi}\right)\left(\varepsilon_0 q_1 p_3 + q_2 p_1\right)\frac{\lambda_0 z_0}{k\pi} \exp\left(-\frac{i\pi l \delta_1}{z_0}\right) - \\
&-\left(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 z_0}{k\pi}\right)\left(q_2 p_2 + \varepsilon_0 q_1 p_4\right)\frac{\lambda_0 z_0}{k\pi} \exp\left(\frac{i\pi l \delta_1}{z_0}\right) + \\
&+\varepsilon_0\left(q_1 + \frac{\lambda_0 z_0}{k\pi} q_2\right)\left(p_1 p_4 - p_2 p_3\right) \exp\left[i\left(\frac{\omega}{v} - \frac{k\pi}{z_0}\right)l\right]\Big\}\Delta^{-1}, \\
G_2 = \Big\{&-\left(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 z_0}{k\pi}\right)\left(\varepsilon_0 q_1 p_4 + q_2 p_2\right) \exp\left[i\left(\frac{\omega}{v} - \frac{k\pi}{z_0} - \frac{\pi\delta_1}{z_0}\right)l\right] + \\
&+\left(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 z_0}{k\pi}\right)\left(q_2 p_1 + \varepsilon_0 q_1 p_3\right) \exp\left[i\left(\frac{\omega}{v} - \frac{k\pi}{z_0} + \frac{\pi\delta_1}{z_0}\right)l\right] + \\
&+\varepsilon_0\left(p_1 p_4 - p_2 p_3\right)\left(q_1 - \frac{\lambda_0 z_0}{k\pi} q_2\right)\Big\}\frac{\lambda_0 z_0}{k\pi}\Delta^{-1} \exp\left[-i\left(\lambda_0 - \frac{k\pi}{z_0}\right)l\right],
\end{aligned}
\tag{56}
$$

$$G_3 = \Big\{ \Big(q_1 - \frac{\lambda_0 \varepsilon_0}{k\tau} q_2\Big)\Big(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big(-i \frac{\pi l \delta_1}{\varepsilon_0}\Big) - $$
$$ - \Big(q_1 + \frac{\lambda_0 \varepsilon_0}{k\tau} q_2\Big)\Big(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big[i\Big(\frac{\omega}{v} - \frac{k\tau}{\varepsilon_0}\Big) l\Big]\Big\} \Delta^{-1},$$
$$G_4 = \Big\{ -\Big(q_1 - \frac{\lambda_0 \varepsilon_0}{k\tau} q_2\Big)\Big(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big(i \frac{\pi l \delta_1}{\varepsilon_0}\Big) + $$
$$ + \Big(q_1 + \frac{\lambda_0 \varepsilon_0}{k\tau} q_2\Big)\Big(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big[i\Big(\frac{\omega}{v} - \frac{k\tau}{\varepsilon_0}\Big) l\Big]\Big\} \Delta^{-1}, \quad (56) \text{ Contd}$$

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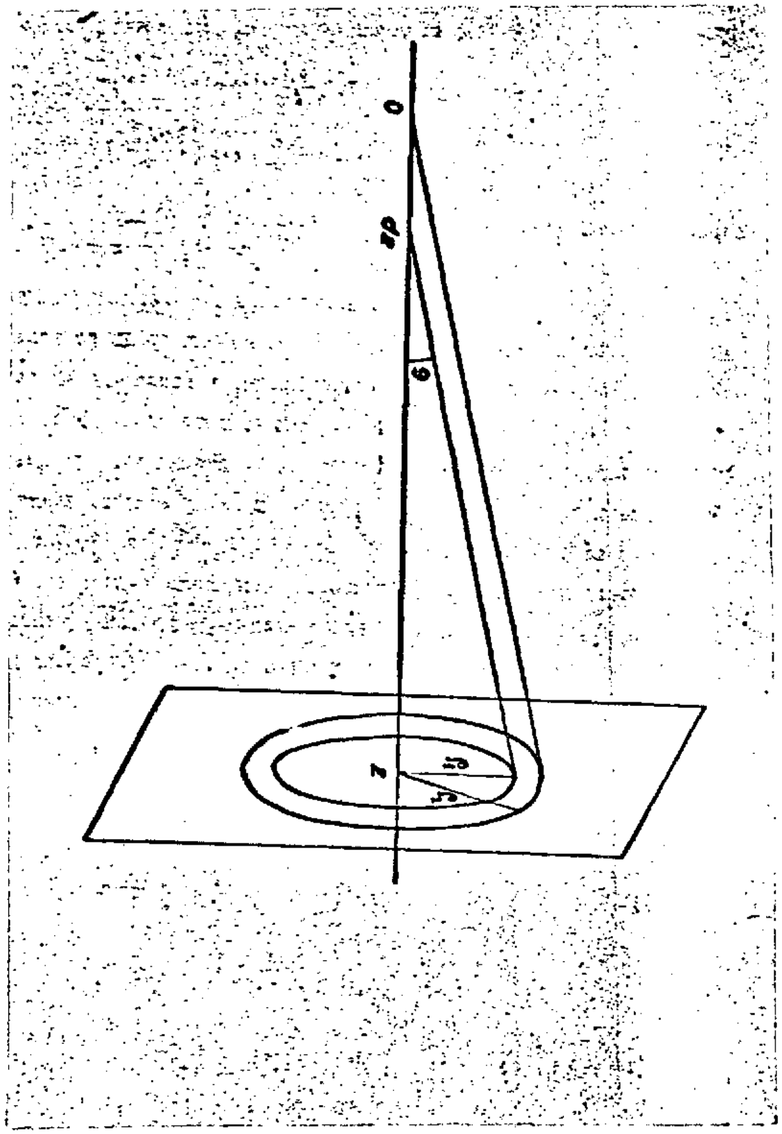
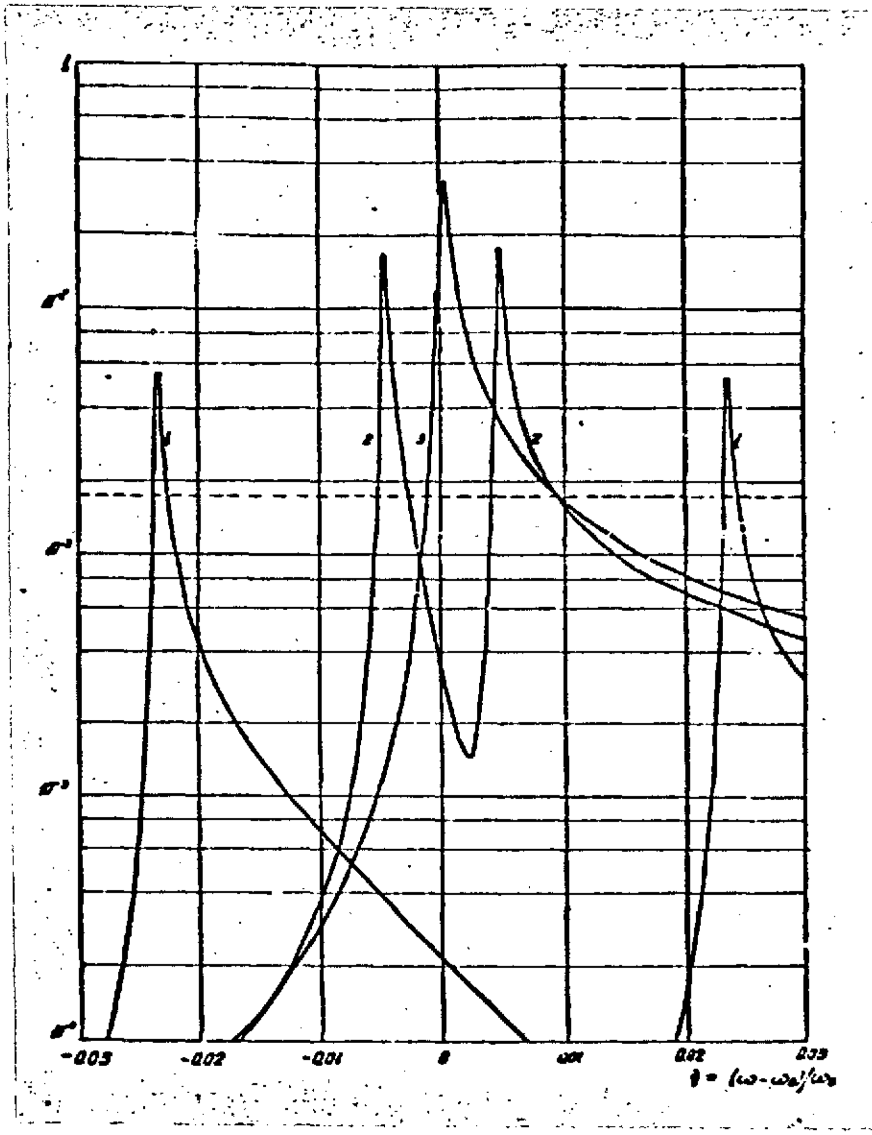
where

$$\Delta = \Big(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big)\Big(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big(-i \frac{\pi l \delta_1}{\varepsilon_0}\Big) - $$
$$ - \Big(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big)\Big(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 \varepsilon_0}{k\tau}\Big) \exp\Big(i \frac{\pi l \delta_1}{\varepsilon_0}\Big). \quad (57)$$

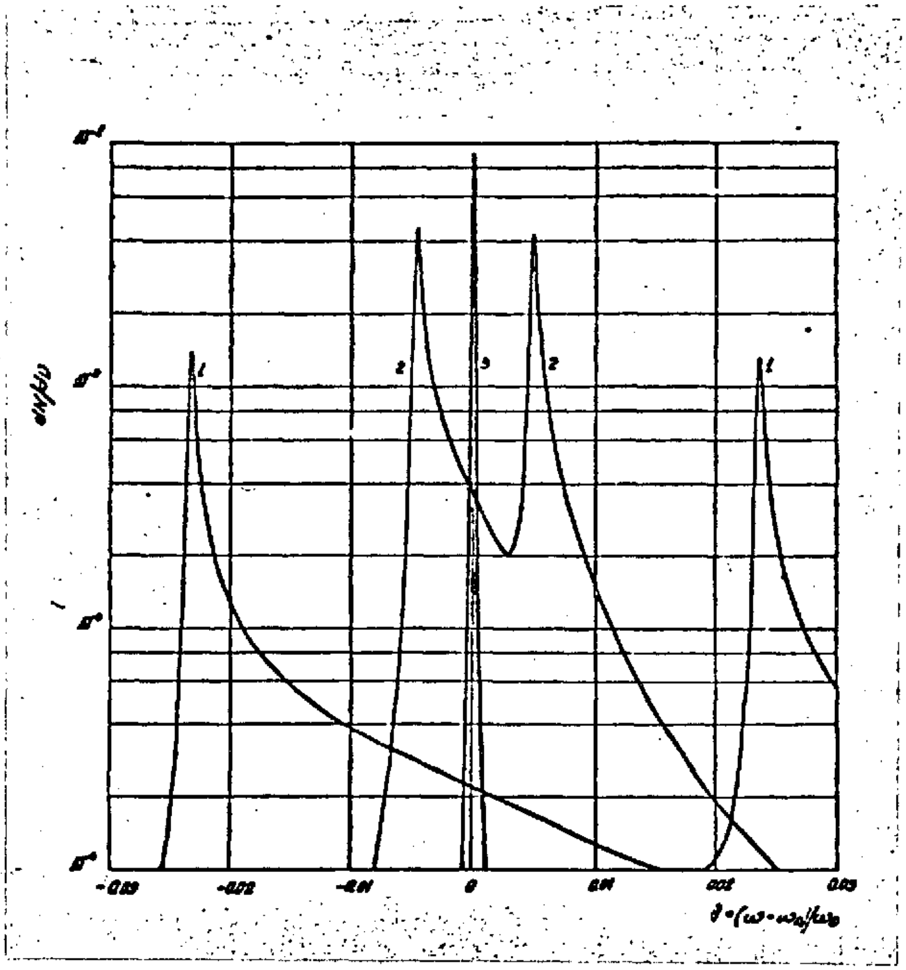
Analyzing Formulas (56), we find that radiation in a vacuum represents the superposition of transitional radiation which arises at the boundaries of the medium and the vacuum and transitional radiation which arises at the medium periodic inhomogeneities over the entire trajectory of motion of a charged particle within the medium, as well as Cherenkov radiation (if it occurs). When the last two types of radiation leave a periodic medium, they naturally undergo reflection and refraction, if the average dielectric constant ε_0 of the medium is not close to unity.

An analysis of the formulas obtained shows that just as was done in [2] the radiation maxima close to the Bragg frequencies, which were found in the preceding section, only occur if the medium is sufficiently extended when $|q\sqrt{a_{1l} a_{-1l}}\, l k/\varepsilon_0| \gtrsim 1$. In this case, these maxima will also exist in a vacuum, both beyond the periodic medium and before it.





1
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
$\xi = (\omega - \omega_0)/\omega_0$



1
2
3
2
1
-0.03
-0.02
-0.01
0
0.01
0.02
0.03

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